

# UNIFORM BOUNDEDNESS OF RATIONAL POINTS AND PREPERIODIC POINTS

BJORN POONEN

**ABSTRACT.** We ask questions generalizing uniform versions of conjectures of Mordell and Lang and combining them with the Morton–Silverman conjecture on preperiodic points. We prove a few results relating different versions of such questions.

## 1. RATIONAL POINTS

**1.1. Uniform boundedness questions.** Our goal is to pose some questions about variation of the number of rational solutions in a family of polynomial equations. The most elementary question of this type we pose is the following:

**Question 1.1.** For each  $n \geq 1$ , is there a number  $B_n$  such that for every  $f \in \mathbb{Q}[x_1, \dots, x_n]$  of total degree 4 such that  $f(x_1, \dots, x_n) = 0$  has finitely many rational solutions, the number of solutions is less than or equal to  $B_n$ ?

It will eventually turn out that already this question is equivalent to much more general questions, namely the number field cases of Questions 1.2 and 1.3 (see Propositions 1.6 and 1.7). In particular, a positive answer for degree 4 would imply a positive answer for arbitrary degree.

To motivate such questions, let us review some uniform boundedness questions in the literature. If  $X$  is a curve of genus  $g > 1$  over a number field  $k$ , then  $X(k)$  is finite [Fal83]. Caporaso, Harris, and Mazur [CHM97] asked whether for each  $g > 1$  and each  $k$ , there is a constant  $B_{g,k}$  such that  $\#X(k) \leq B_{g,k}$  for all  $X$  of genus  $g$  over  $k$ . The answer is unknown even for  $g = 2$  and  $k = \mathbb{Q}$ . Caporaso, Harris, and Mazur proved that a positive answer would follow from the Bombieri–Lang conjecture that the  $k$ -rational points on a positive-dimensional variety of general type are not Zariski dense. Pacelli [Pac97] generalized this to show that the Bombieri–Lang conjecture implies that the constant  $B_{g,k}$  can be chosen to depend only on  $g$  and  $[k : \mathbb{Q}]$ ; this would imply its generalization to finitely generated field extensions of  $\mathbb{Q}$  (i.e., function fields of varieties over number fields), because a curve over a field  $k$  of degree  $d$  over  $\mathbb{Q}(t_1, \dots, t_n)$  can be specialized to a curve of the same genus over a number field of degree  $d$  over  $\mathbb{Q}$  having at least as many points. Such results were generalized to higher-dimensional varieties for which all subvarieties are of general type: see [AV96, Abr97].

Our main question generalizes such questions to arbitrary families of varieties:

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**Question 1.2.** Let  $k$  be a finitely generated extension of  $\mathbb{Q}$ . Let  $\pi: X \rightarrow S$  be a morphism of finite-type  $k$ -schemes. For  $s \in X(k)$ , let  $X_s$  be the fiber  $\pi^{-1}(s)$ . Must  $\{\#X_s(k) : s \in S(k)\}$  be finite?

If some  $\#X_s(k)$  is infinite, that is OK: it contributes just the one element  $\aleph_0$  to the set whose finiteness is in question. So the question is really about the uniform boundedness of  $\#X_s(k)$  for the  $s \in S(k)$  for which  $X_s(k)$  is finite.

We generalize further by considering points  $s$  over finite extensions  $L$  of fixed (or bounded) degree over  $k$ :

**Question 1.3.** Fix  $k$  and  $\pi: X \rightarrow S$  as in Question 1.2. Let  $D \geq 1$ . Must  $\{\#X_s(L) : [L : k] = D, s \in S(L)\}$  be finite?

## 1.2. Variants.

**Question 1.4.** Under the hypotheses of Question 1.3, let  $z_s \in \mathbb{Z}_{\geq 0}$  be the number of irreducible components of the Zariski closure of  $X_s(L)$  in  $X_s$ . Must  $\{z_s : [L : k] = D, s \in S(L)\}$  be finite?

Given a finite-type  $\mathbb{Q}$ -scheme  $X$ , and a subset  $A \subset X(\mathbb{Q})$ , let  $\overline{A}$  be the closure of  $A$  in  $X(\mathbb{R})$  with respect to the Euclidean topology. Mazur [Maz92] conjectured that the set of connected components of the topological space  $\overline{X(\mathbb{Q})}$  is finite for every  $X$ .

**Question 1.5.** Under the hypotheses of Question 1.2, but with  $k = \mathbb{Q}$ , let  $c_s$  be the number of connected components of  $\overline{X_s(\mathbb{Q})}$ . Must  $\{c_s : s \in S(\mathbb{Q})\}$  be finite?

**1.3. Implications.** Every finite-type  $k$ -scheme is a finite union of finite-type affine  $k$ -schemes, so each of Questions 1.2, 1.4, and 1.5 may be reduced to the case where  $S$  and  $X$  are affine.

Question 1.4 is stronger than Question 1.3. Question 1.5 is stronger than the  $k = \mathbb{Q}$  case of Question 1.2. Less trivial is the following:

**Proposition 1.6.** *For each finitely generated extension  $k$  of  $\mathbb{Q}$ , Questions 1.2 and 1.3 are equivalent.*

*Proof.* Question 1.2 is the  $D = 1$  case of Question 1.3.

For the reduction in the opposite direction, fix an instance  $\pi: X \rightarrow S$  of Question 1.3. We may assume that  $X$  and  $S$  are affine. View

$$T := \operatorname{Spec} \frac{k[a_{D-1}, \dots, a_0, t]}{(t^D + a_{D-1}t^{D-1} + \dots + a_0)},$$

as a finite scheme over  $\mathbb{A}^D = \operatorname{Spec} k[a_{D-1}, \dots, a_0]$ . The restrictions of scalars  $\mathcal{X} := \operatorname{Res}_{T/\mathbb{A}^D}(X \times_k T)$  and  $\mathcal{S} := \operatorname{Res}_{T/\mathbb{A}^D}(S \times_k T)$  exist [BLR90, 7.6, Theorem 4], and  $\pi$  induces  $\Pi: \mathcal{X} \rightarrow \mathcal{S}$ . If  $a \in \mathbb{A}^D(k)$ , its fiber in  $T$  defines a finite  $k$ -algebra  $L$  (not necessarily a field); then each point  $s' \in \mathcal{S}(k)$  mapping to  $a$  corresponds to a point  $s \in S(L)$ , and the fiber  $\Pi^{-1}(s')$  equals  $\operatorname{Res}_{L/k}(X_s)$ , whose  $k$ -points are in bijection with  $X_s(L)$ . Moreover, every degree- $D$  field extension  $L$  of  $k$  arises from some  $a \in \mathbb{A}^D(k)$ . Thus a positive answer to Question 1.2 for  $\Pi$  would yield a positive answer to Question 1.3 for  $\pi$ .  $\square$

**Proposition 1.7.** *Question 1.2 for number fields is equivalent to Question 1.1.*

*Proof.* Applying Question 1.2 with  $k = \mathbb{Q}$  and  $X \rightarrow S$  the universal family of degree 4 hypersurfaces in  $\mathbb{A}^n$  yields Question 1.1.

Now consider the opposite direction. The proof of Proposition 1.6 shows that the Question 1.2 for number fields is equivalent to Question 1.2 for  $\mathbb{Q}$ . The latter can be reduced to the case where  $X$  and  $S$  are affine. To complete the proof, we show that each  $X_s$  has the same number of rational points as a certain affine hypersurface of degree 4 in  $\mathbb{A}^n$  for some  $n$  depending only on  $X \rightarrow S$ : each polynomial in the system defining  $X_s$  can be rewritten as a system of equations of degree at most 2 by Skolem's trick of introducing new indeterminates to represent the results of intermediate steps in a calculation of a polynomial, and the union of these systems can be collapsed into a single polynomial by taking the sum of squares; if necessary, add in  $z^4$  for a new indeterminate  $z$  to ensure that the polynomial is of degree exactly 4. The number of indeterminates used in this rewriting of  $X_s$  is uniform in  $s$ .  $\square$

**1.4. Counterexamples.** Question 1.2 has a negative answer for some finitely generated fields of characteristic  $p > 0$ . For example, if  $k := \mathbb{F}_p(t)$  for some  $p > 2$ , then in the family of non-smooth curves  $X_a: x - ax^p = y^p$ , the members with  $a \in k - k^p$  have only finitely many  $k$ -points, but their number is unbounded as  $a$  varies [AV96, Theorem 4.1]. For another family, this time consisting of smooth curves, see [CUV12].

We do not know of “natural” fields of characteristic 0 for which the answer to Question 1.2 is negative, but we can artificially construct such fields:

**Proposition 1.8.** *There exists a countable field  $k$  of characteristic 0 for which Question 1.2 has a negative answer.*

*Proof.* Let  $X_1, X_2, \dots$  be representatives for the isomorphism classes of the smooth projective geometrically integral curves of genus 2 over  $\mathbb{Q}$ . Let  $Y_i := X_1 \times X_2^2 \times \dots \times X_i^i$ . Let  $K_i$  be the function field of  $Y_i$ . The projection  $Y_{i+1} \rightarrow Y_i$  induces an injection  $K_i \hookrightarrow K_{i+1}$ . Let  $k = \varinjlim K_i$ .

For each  $i$ , composing any of the  $i$  projections  $Y_i \rightarrow X_i$  with an automorphism of  $X_i$  yields an element of  $X_i(Y_i) = X_i(K_i) \subseteq X_i(k)$ . Since any nonconstant morphism between genus 2 curves in characteristic 0 is an isomorphism and the automorphism group of a genus 2 curve is finite, any nonconstant morphism  $Y_j \rightarrow X_i$  for  $j \geq i$  factors through one of the projections  $Y_j \rightarrow X_i$  and hence corresponds to an already-constructed point of  $X_i(k)$ .

Thus  $i \leq \#X_i(k) < \infty$  for each  $i$ , so Question 1.2 for a versal family of genus 2 curves over  $k$  has a negative answer.  $\square$

## 2. TORSION POINTS ON ABELIAN VARIETIES

If  $A$  is an abelian variety over a number field  $k$ , then the torsion subgroup  $A(k)_{\text{tors}}$  is finite: this is a small part of the Mordell–Weil theorem [Wei29], and can be proved using height functions or  $p$ -adic methods. This suggests the following well-known question:

**Question 2.1.** Is there a bound on  $\#A(k)_{\text{tors}}$  depending only on  $\dim A$  and  $[k : \mathbb{Q}]$ ?

For  $\dim A = 1$ , the answer is yes [Maz77, KM95, Mer96]. For  $\dim A > 1$ , there are only partial results: see [CT11], which also considers the geometric analogue. If the answer is yes, then the answer is yes also over finitely generated extensions  $k$  of  $\mathbb{Q}$ : restriction of scalars lets us reduce to the case  $k = \mathbb{Q}(t_1, \dots, t_n)$ , and then specialization lets us remove one indeterminate at a time without enlarging the torsion subgroup.

*Remark 2.2.* Uniform boundedness for the number of rational points on curves of genus  $g > 1$  over a finitely generated extension  $k$  of  $\mathbb{Q}$  for each  $g$  and  $k$  would imply a positive answer to Question 1.2 for all families  $X \rightarrow S$  whose fibers are of dimension at most 1. Indeed, the geometry of the singularities and 0-dimensional components of the fibers is uniformly bounded, and irreducible components that are not geometrically irreducible have rational points constrained to the singular locus, so it would suffice to prove uniform boundedness of  $C(k)$  for smooth projective geometrically integral curves  $C$  of bounded genus with finitely many  $k$ -points. If  $C$  has genus 0, then  $C(k)$  is empty or infinite. If  $C$  has genus 1, then  $C(k)$  is empty, infinite, or of cardinality bounded by the previous paragraph. If  $C$  has genus greater than 1, then our hypothesis applies.

### 3. PREPERIODIC POINTS

Given a morphism  $f: X \rightarrow X$  of  $k$ -schemes, a point in  $X(k)$  is called **preperiodic** if its forward trajectory is finite; let  $\text{PrePer}(f, k)$  be the set of such points. Northcott [Nor50] invented the theory of height functions to prove that if  $k$  is a number field and  $f: \mathbb{P}^n \rightarrow \mathbb{P}^n$  is a morphism of degree  $d \geq 2$  over  $k$ , then  $\text{PrePer}(f, k)$  is finite. The Morton–Silverman conjecture [MS94, p. 100] predicts that  $\#\text{PrePer}(f, k)$  is bounded by a constant depending only on  $n$ ,  $d$ , and  $[k : \mathbb{Q}]$ .

*Remark 3.1.* Morton and Silverman observed that applying their conjecture to the morphism  $\mathbb{P}^1 \rightarrow \mathbb{P}^1$  induced by multiplication-by-2 on the  $x$ -coordinate of an elliptic curve  $A$  yields uniform boundedness of torsion points on elliptic curves over number fields. By [Fak03, Corollary 2.4], the Morton–Silverman conjecture also implies a positive answer to Question 2.1 for abelian varieties of arbitrary dimension.

We may now ask the analogue of Question 1.3 for rational preperiodic points:

**Question 3.2.** Let  $k$  be a finitely generated extension of  $\mathbb{Q}$ . Let  $\pi: X \rightarrow S$  be a morphism of finite-type  $k$ -schemes. Let  $f: X \rightarrow X$  be an  $S$ -morphism. If  $L$  is a finite extension of  $k$  and  $s \in S(L)$ , let  $X_s := \pi^{-1}(s)$  and let  $f_s: X_s \rightarrow X_s$  be the restriction of  $f$  to  $X_s$ . Let  $D \geq 1$ . Must  $\{\#\text{PrePer}(f_s, L) : [L : k] = D, s \in S(L)\}$  be finite?

**3.1. Variants.** Questions 1.4 and 1.5 also admit analogues in which  $X_s(L)$  is replaced by  $\text{PrePer}(f_s, L)$ .

**3.2. Implications.** Taking  $f = \text{id}$  in Question 3.2 yields Question 1.3.

Question 3.2 for the universal family of degree- $d$  self-maps  $\mathbb{P}^n \rightarrow \mathbb{P}^n$  is equivalent to the Morton–Silverman conjecture, so by Remark 3.1, a positive answer to Question 3.2 would imply a positive answer to Question 2.1. In fact, a positive answer to Question 3.2 also implies a positive answer to Question 2.1 directly: use Zarhin’s trick [Zar74; Mil86, Remark 16.12] to reduce to the case of principally polarized abelian varieties of a fixed dimension (8 times as large), for which a versal family  $\mathcal{A} \rightarrow S$  exists, and then apply Question 3.2 to the  $S$ -morphism  $[2]: \mathcal{A} \rightarrow \mathcal{A}$ .

As in Section 1.3, Question 3.2 can be reduced to the case in which  $S$  is affine, but it is not clear whether we can assume also that  $X$  is affine, since  $X$  might not be a union of  $f$ -stable affine subschemes. Because of this, the analogue of Proposition 1.6 for preperiodic points is weakened slightly to ensure that the restrictions of scalars in its proof exist without first making  $X$  affine:

**Proposition 3.3.** *Let  $k$  be a finitely generated extension of  $\mathbb{Q}$ . If the answer to Question 3.2 for quasi-projective schemes over  $k$  is positive for  $D = 1$ , then it is positive also for arbitrary  $D$ .*

*Proof.* Let  $\pi: X \rightarrow S$  and  $f: X \rightarrow X$  be an instance of Question 3.2 for a given  $k$  and  $D$ . We can no longer assume that  $X$  is affine, but since  $X$  and  $S$  are quasi-projective, [BLR90, 7.6, Theorem 4] still applies to let us construct  $\Pi: \mathcal{X} \rightarrow \mathcal{S}$  as in the proof of Proposition 1.6, and we also obtain an  $\mathcal{S}$ -morphism  $F: \mathcal{X} \rightarrow \mathcal{X}$ . Each  $s' \in \mathcal{S}(k)$ , corresponds to a finite  $k$ -algebra  $L$  with a point  $s \in S(L)$ , and  $\text{PrePer}(F_{s'}, k) \subseteq \mathcal{X}_{s'}(k)$  corresponds to  $\text{PrePer}(f_s, L) \subseteq X_s(L)$ . So a positive answer to Question 3.2 for  $(\Pi, F, 1)$  would yield a positive answer for  $(\pi, f, D)$ .  $\square$

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DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MA 02139-4307, USA

*E-mail address:* `poonen@math.mit.edu`

*URL:* `http://math.mit.edu/~poonen/`